

SUPPLEMENT TO “SALVAGING FALSIFIED
INSTRUMENTAL VARIABLE MODELS”
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IN THIS SUPPLEMENT, we extend our analysis of the linear instrumental variable model to allow for multiple endogenous variables.

S1. THE FF AND FAS FOR MULTIPLE ENDOGENOUS VARIABLES

Theorem 1 characterizes the identified set for the vector of coefficients on the endogenous variables, as a function of the exclusion restriction relaxation. Our subsequent characterizations of the falsification frontier and the falsification adaptive set, however, restricted attention to the case with just one endogenous variable; see Proposition 2 and Theorem 2. In this section, we extend these two results to the general case with $K \geq 1$ endogenous variables. These results can also be used if, for example, a single endogenous variable has interactions with covariates or if the outcome equation is nonlinear in this variable.

In this general case we assume all instruments are relevant for simplicity. To state our new assumption, we consider submatrices of Π . Let $\mathcal{L} \subseteq \{1, \dots, L\}$. Let $\Pi_{\mathcal{L}}$ be the $|\mathcal{L}| \times K$ submatrix of Π formed by removing all rows $\ell \notin \mathcal{L}$. Let π'_{ℓ} denote the ℓ th row of the matrix Π . We strengthen and generalize assumption A1 as follows.

ASSUMPTION A1'—Relevance: *The following hold:*

1. For all $\mathcal{L} \subseteq \{1, \dots, L\}$ with $|\mathcal{L}| = K$, $\Pi_{\mathcal{L}}$ has full rank.
2. For all $\mathcal{L} \subseteq \{1, \dots, L\}$ such that $|\mathcal{L}| = K + 1$, $\{\pi_{\ell} : \ell \in \mathcal{L}\}$ are affinely independent. That is, for all $\mathcal{L} = \{\ell_1, \dots, \ell_{K+1}\}$,

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ \pi_{\ell_1} & \pi_{\ell_2} & \dots & \pi_{\ell_{K+1}} \end{pmatrix}$$

has full rank.

A1.1' implies that any set \mathcal{L} of K instruments uniquely define the coefficients $\beta_{\mathcal{L}}^{2SLS} = \Pi_{\mathcal{L}}^{-1} \psi_{\mathcal{L}}$, where $\psi_{\mathcal{L}}$ equals the subvector of ψ after removing all components $\ell \notin \mathcal{L}$. $\beta_{\mathcal{L}}^{2SLS}$ is the population 2SLS coefficient on X using $Z_{\mathcal{L}}$ as excluded instruments and $Z_{-\mathcal{L}}$ as controls. Here we partition $Z = (Z_{\mathcal{L}}, Z_{-\mathcal{L}})$ based on the indices in \mathcal{L} .

A1.2' means that there does not exist a hyperplane that passes through all of the π_{ℓ} vectors. It is equivalent to linear independence of $(\pi_L - \pi_1, \dots, \pi_2 - \pi_1)$.

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For $|\mathcal{L}| = K + 1$, let

$$\mathcal{P}_{\mathcal{L}} = \text{conv}(\{\beta_{\mathcal{L} \setminus \{\ell\}}^{2\text{SLS}} : \ell \in \mathcal{L}\}).$$

This is the convex hull of $K + 1$ just-identified 2SLS estimands in \mathbb{R}^K . We show that the falsification frontier and the falsification adaptive set can be constructed from $\mathcal{P}_{\mathcal{L}}$.

PROPOSITION S1: *Suppose A1', A2, and A3 hold. Suppose the joint distribution of (Y, X, Z) is known. Then the falsification frontier is the set*

$$\text{FF} = \{\delta \in \mathbb{R}_{\geq 0}^L : \delta_{\ell} = |\psi_{\ell} - \pi'_{\ell} b|, b \in \mathcal{P}_{\mathcal{L}}, \mathcal{L} \subseteq \{1, \dots, L\}, |\mathcal{L}| = K + 1\}.$$

THEOREM S1: *Suppose A1', A2, and A3 hold. Suppose the joint distribution of (Y, X, Z) is known. Let*

$$\mathcal{P} = \bigcup_{\mathcal{L} \subseteq \{1, \dots, L\}: |\mathcal{L}|=K+1} \mathcal{P}_{\mathcal{L}}.$$

Then \mathcal{P} is the falsification adaptive set.

Like the $K = 1$ case (Theorem 2), \mathcal{P} can be computed by running a variety of 2SLS regressions. Unlike that case, however, \mathcal{P} is generally not convex, even though each $\mathcal{P}_{\mathcal{L}}$ is convex. Nonetheless, we are often only interested in linear functionals of the coefficient vector β . For example, we often care about just one component of β . The following corollary shows that the falsification adaptive set for a linear functional of β again has a simple form. For this result, let

$$\text{FAS}^* = \text{conv}(\{\beta_{\mathcal{L}}^{2\text{SLS}} : \mathcal{L} \subseteq \{1, \dots, L\}, |\mathcal{L}| = K\}) \quad (\text{S1})$$

denote the convex hull of the set of all just-identified 2SLS estimands.

COROLLARY S1: *Suppose A1', A2, and A3 hold. Suppose the joint distribution of (Y, X, Z) is known. Then FAS^* contains the falsification adaptive set for β . Moreover, for any $\alpha \in \mathbb{R}^K$ the falsification adaptive set for $\alpha' \beta$ is*

$$\left[\min_{\mathcal{L} \subseteq \{1, \dots, L\}: |\mathcal{L}|=K} \alpha' \beta_{\mathcal{L}}^{2\text{SLS}}, \max_{\mathcal{L} \subseteq \{1, \dots, L\}: |\mathcal{L}|=K} \alpha' \beta_{\mathcal{L}}^{2\text{SLS}} \right].$$

This result shows that the FAS characterized in Theorem S1 is contained in the simpler set FAS^* . It also shows that we can simply cycle through all possible just-identified models, compute the corresponding 2SLS estimand, take the convex hull, and project it onto one component to get the FAS for that component.

To illustrate these results, consider the two endogenous variables ($K = 2$) and three instruments ($L = 3$) case. Hence we have $L = K + 1$. Consider the left plot in Figure S1. This plot shows possible values (b_1, b_2) of the coefficients on X . The exclusion restriction from instrument ℓ imposes a single linear constraint $\psi_{\ell} = \pi'_{\ell} b$. These constraints are simply lines in \mathbb{R}^2 . Since there are three instruments, there are three constraints. When these three lines do not intersect at a common point, the baseline model is falsified. This case is shown in the figure. Suppose we drop the exclusion restriction for instrument ℓ . Then two linear constraints remain, β is point identified, and it equals the intersection point $\beta_{\{1, 2, 3\} \setminus \{\ell\}}^{2\text{SLS}}$. Repeating this for $\ell \in \{1, 2, 3\}$ and taking the convex hull yields the falsification adaptive set, which is shown as the shaded triangular region.

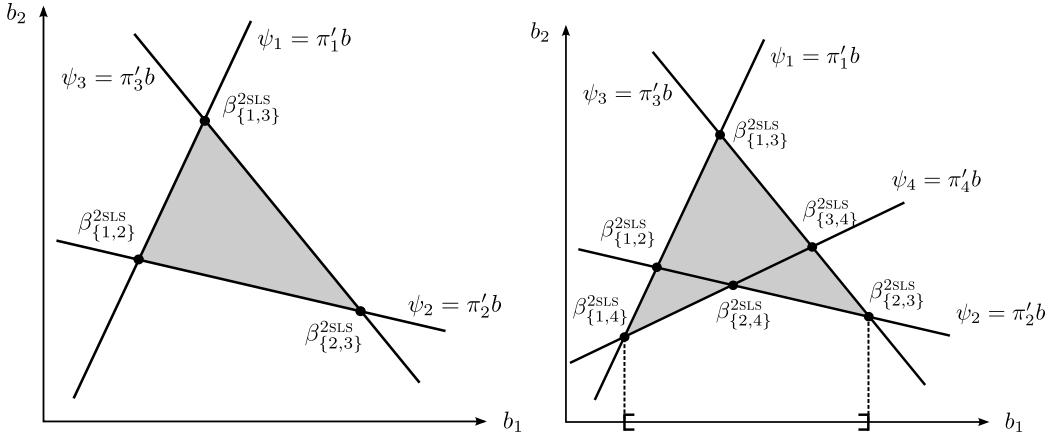


FIGURE S1.—Example with $K = 2$ endogenous variables. Left: $L = 3$ instruments. Right: $L = 4$ instruments. In both plots, the falsification adaptive set for (β_1, β_2) is the shaded region. In the right plot, the falsification adaptive set for β_1 is shown as the projection onto the first component.

The right plot in Figure S1 illustrates the $L > K + 1$ case. Here we have $K = 2$ and $L = 4$. There are 6 different just-identified 2SLS estimands. The falsification adaptive set is no longer convex. Nonetheless, the projection of the convex hull of all just-identified 2SLS estimands onto the first component gives the falsification adaptive set for β_1 . Moreover, this projection can be computed by simply taking the largest and smallest values of β_1 among the just-identified 2SLS estimands, as shown in Corollary S1.

S2. PROOFS FOR SECTION S1

We next present a sequence of lemmas that lead to the proofs of the results in Section S1. Here we omit proofs of some of the more straightforward lemmas, but full proofs are available in Appendix K of Masten and Poirier (2020).

We begin by showing a basic geometric fact about the set FAS^* when $L = K + 1$. Here and elsewhere we use the notation $\beta_{-\ell}^{2\text{SLS}} = \beta_{\{1, \dots, L\} \setminus \{\ell\}}^{2\text{SLS}}$.

LEMMA S1: *Suppose A1', A2, and A3 hold. Suppose $L = K + 1$. Then exactly one of the following holds:*

1. $\beta_{-\ell}^{2\text{SLS}} = \beta_{-\ell'}^{2\text{SLS}}$ for all $\ell, \ell' \in \{1, \dots, L\}$.
2. $\pi'_\ell \beta_{-\ell}^{2\text{SLS}} \neq \psi_\ell$ for all $\ell \in \{1, \dots, L\}$.

The next lemma shows that, when $L = K + 1$ and FAS^* is not a singleton, we can write any element of \mathbb{R}^K as a weighted sum of our just-identified 2SLS estimands.

LEMMA S2: *Suppose A1', A2, and A3 hold. Suppose $L = K + 1$. Assume that FAS^* is not a singleton. Then for any $b \in \mathbb{R}^K$ there exist weights $w_1(b), \dots, w_L(b)$ such that $\sum_{\ell=1}^L w_\ell(b) = 1$ and*

$$b = \sum_{\ell=1}^L w_\ell(b) \beta_{-\ell}^{2\text{SLS}}.$$

Define $\delta_\ell(b) = |\psi_\ell - \pi'_\ell b|$ for all $\ell = 1, \dots, L$. Let $\delta(b) = (\delta_1(b), \dots, \delta_L(b))$. We next show that, in the $L = K + 1$ case, the identified set for β is a singleton for $\delta = \delta(b)$, and $b \in \text{FAS}^*$.

LEMMA S3: *Suppose A1', A2, and A3 hold. Suppose $L = K + 1$. Let $b \in \text{FAS}^*$. Then $\mathcal{B}(\delta(b)) = \{b\}$.*

PROOF OF LEMMA S3: By Lemma S1, there are two cases to consider: FAS^* is either a singleton or a nondegenerate simplex in \mathbb{R}^K .

Case 1. Suppose $\text{FAS}^* = \{b\}$ is a singleton. By the definition of FAS^* , this implies that $b = \beta_{\mathcal{L}}^{\text{2SLS}}$ for any $\mathcal{L} \subseteq \{1, \dots, L\}$ with $|\mathcal{L}| = K$. It also implies $\delta(b) = 0_L$. Therefore $\mathcal{B}(0_L) = \bigcap_{\ell=1}^L B_\ell(0) = \bigcap_{\mathcal{L}: |\mathcal{L}|=K} (\bigcap_{\ell \in \mathcal{L}} B_\ell(0)) = \{b\}$ by $\bigcap_{\ell \in \mathcal{L}} B_\ell(0) = \{b\}$.

Case 2. Suppose FAS^* is not a singleton. Then $\pi'_\ell \beta_{-\ell}^{\text{2SLS}} \neq \psi_\ell$ for all $\ell \in \{1, \dots, L\}$. We prove equality of sets by showing that both directions of set inclusion hold.

Step 1 (\supseteq). First we show that $\mathcal{B}(\delta(b)) \supseteq \{b\}$. By definition of $\delta_\ell(\cdot)$, $\psi_\ell - \pi'_\ell b \in [-\delta_\ell(b), \delta_\ell(b)]$ for all ℓ . Thus, by the characterization of $\mathcal{B}(\cdot)$ in Theorem 1, $b \in \mathcal{B}(\delta(b))$.

Step 2 (\subseteq). Next we show that $\mathcal{B}(\delta(b)) \subseteq \{b\}$. First suppose $\delta(b) = 0_L$. In this case the baseline model is not falsified and FAS^* is a singleton. This is a contradiction. So we must have $\delta_\ell(b) > 0$ for some ℓ .

We will show that any element $b^* \neq b$ is not in $\mathcal{B}(\delta(b))$. The set FAS^* is a polytope. Consider its alternative half-space representation. The half-spaces correspond to one side of the hyperplanes $B_\ell(0)$. Formally, write

$$\text{FAS}^* = \bigcap_{\ell=1}^L \{\tilde{b} \in \mathbb{R}^K : \psi_\ell - \pi'_\ell \tilde{b} \leq 0\}. \quad (\text{S2})$$

We assume without loss of generality that all L inequalities go in the same direction. This is because $\psi_\ell - \pi'_\ell \tilde{b} \geq 0$ can be rewritten as $-\psi_\ell - (-\pi'_\ell \tilde{b}) \leq 0$, which is equivalent to replacing instrument Z_ℓ with $-Z_\ell$. Neither the estimands $\beta_{\mathcal{L}}^{\text{2SLS}}$ nor the set FAS^* are affected by these sign normalizations.

Noting that $\mathcal{B}(\delta)$ is an intersection of half-spaces and evaluating it at $\delta(b)$ gives

$$\begin{aligned} \mathcal{B}(\delta(b)) &= \bigcap_{\ell=1}^L \{\tilde{b} \in \mathbb{R}^K : -\delta_\ell(b) \leq \psi_\ell - \pi'_\ell \tilde{b} \leq \delta_\ell(b)\} \\ &= \left(\bigcap_{\ell=1}^L \{\tilde{b} \in \mathbb{R}^K : \psi_\ell - \pi'_\ell \tilde{b} \geq -|\psi_\ell - \pi'_\ell b|\} \right) \\ &\quad \cap \left(\bigcap_{\ell=1}^L \{\tilde{b} \in \mathbb{R}^K : \psi_\ell - \pi'_\ell \tilde{b} \leq |\psi_\ell - \pi'_\ell b|\} \right) \\ &\equiv \mathcal{P}_1(b) \cap \mathcal{P}_2(b). \end{aligned}$$

To complete this proof, it suffices to show $b^* \notin \mathcal{P}_1(b)$. By Lemma S2, any element in \mathbb{R}^K can be written as a linear combination of the L different just-identified 2SLS estimands.

In particular, we can write b^* in this way:

$$b^* = \sum_{\ell=1}^L w_{\ell}(b^*) \beta_{-\ell}^{2SLS},$$

where $w_{\ell}(b^*)$ are weights that sum to one, $\sum_{\ell=1}^L w_{\ell}(b^*) = 1$.

Since $b \in \text{FAS}^*$, $\psi_{\ell} - \pi'_{\ell} b \leq 0$ for all ℓ . This follows directly from our half-space representation of FAS^* . Thus $-|\psi_{\ell} - \pi'_{\ell} b| = \psi_{\ell} - \pi'_{\ell} b$ for all ℓ . Hence

$$\mathcal{P}_1(b) = \bigcap_{\ell=1}^L \{\tilde{b} \in \mathbb{R}^K : \psi_{\ell} - \pi'_{\ell} \tilde{b} \geq -|\psi_{\ell} - \pi'_{\ell} b|\} = \bigcap_{\ell=1}^L \{\tilde{b} \in \mathbb{R}^K : \pi'_{\ell}(\tilde{b} - b) \leq 0\}.$$

So $b^* \in \mathcal{P}_1(b)$ if and only if $\pi'_{\ell}(b^* - b) \leq 0$ for all ℓ . Focus on just one ℓ for a moment. Then

$$\begin{aligned} \pi'_{\ell}(b^* - b) &= \sum_{s=1}^L (w_s(b^*) - w_s(b)) \pi'_{\ell} \beta_{-s}^{2SLS} \\ &= \sum_{s \neq \ell} (w_s(b^*) - w_s(b)) \psi_{\ell} + (w_{\ell}(b^*) - w_{\ell}(b)) \pi'_{\ell} \beta_{-\ell}^{2SLS} \\ &= (\psi_{\ell} - \pi'_{\ell} \beta_{-\ell}^{2SLS}) \sum_{s \neq \ell} (w_s(b^*) - w_s(b)). \end{aligned}$$

The first line follows from Lemma S2. The second follows from $\psi_{\ell} = \pi'_{\ell} \beta_{-s}^{2SLS}$ when $s \neq \ell$ by the definition of these 2SLS estimands. The third follows from the difference in weights summing to zero.

Next notice that $\psi_{\ell} - \pi'_{\ell} \beta_{-\ell}^{2SLS} < 0$. This follows from $\beta_{-\ell}^{2SLS} \in \text{FAS}^*$, the half-space representation of FAS^* , and since FAS^* is a nondegenerate simplex. Suppose by way of contradiction that $b^* \in \mathcal{P}_1(b)$. Then $\pi'_{\ell}(b^* - b) \leq 0$ for all ℓ . We have just seen that this implies $\sum_{s \neq \ell} (w_s(b^*) - w_s(b)) \geq 0$ for all ℓ . But now we have

$$0 = \sum_{s=1}^L (w_s(b^*) - w_s(b)) = \sum_{s \neq \ell} (w_s(b^*) - w_s(b)) + (w_{\ell}(b^*) - w_{\ell}(b)).$$

Thus $w_{\ell}(b^*) - w_{\ell}(b) = \sum_{s \neq \ell} (w_s(b) - w_s(b^*)) \leq 0$ for all ℓ . Since $w_{\ell}(b^*) - w_{\ell}(b)$ sums to zero, $w_{\ell}(b^*) = w_{\ell}(b)$ for all ℓ . This implies $b^* = b$, a contradiction. Thus $b^* \notin \mathcal{P}_1(b)$. *Q.E.D.*

LEMMA S4: *Suppose A1', A2, and A3 hold. Suppose $L \geq K + 1$. Let $b \in \mathcal{P}$. Then $\mathcal{B}(\delta(b)) = \{b\}$.*

PROOF OF LEMMA S4: We prove set equality by showing that both directions of set inclusion hold.

Step 1 (\supseteq). The proof of this step from Lemma S3 applies without modification.

Step 2 (\subseteq). Since $b \in \mathcal{P}$ there is some $\mathcal{L} \subseteq \{1, \dots, L\}$ with $|\mathcal{L}| = K + 1$ such that $b \in \mathcal{P}_{\mathcal{L}}$. Let

$$\mathcal{B}_{\mathcal{L}}(\delta) = \bigcap_{\ell \in \mathcal{L}} B_{\ell}(\delta).$$

By Lemma S3, $\mathcal{B}_{\mathcal{L}}(\delta(b)) = \{b\}$. By definition, $\mathcal{B}_{\mathcal{L}}(\delta) \supseteq \mathcal{B}(\delta)$. Thus $\mathcal{B}(\delta(b)) \subseteq \{b\}$. *Q.E.D.*

The following variation on Farkas' Lemma (e.g., Corollary 22.3.1 on p. 200 of Rockafellar (1970); Border (2019) provides an extensive discussion) is helpful.

LEMMA S5—Variation on Farkas' Lemma: *Let $x_1, \dots, x_n \in \mathbb{R}^K$. Then $0_K \notin \text{conv}(\{x_1, \dots, x_n\})$ if and only if there exists a $p \in \mathbb{R}^K$ such that $p'x_i > 0$ for all $i = 1, \dots, n$.*

The next two technical lemmas are used in the proof of Lemma S8, which is then used in the proof of Lemma S9.

LEMMA S6: *Suppose A1' holds. Suppose $L = K + 1$. Suppose FAS* is not a singleton. Then $\psi_{\ell} - \pi'_{\ell}b = w_{\ell}(b)(\psi_{\ell} - \pi'_{\ell}\beta_{-\ell}^{2\text{SLS}})$ for all $\ell = 1, \dots, L$.*

LEMMA S7: *Suppose A1' holds. Suppose $L = K + 1$. Suppose FAS* is not a singleton. Without loss of generality (see equation (S2) and the surrounding discussion), write*

$$\text{FAS}^* = \bigcap_{\ell=1}^L \{b \in \mathbb{R}^K : \psi_{\ell} - \pi'_{\ell}b \leq 0\}.$$

Then there are no $b \in \mathbb{R}^K$ such that $\psi_{\ell} - \pi'_{\ell}b \geq 0$ for all $\ell = 1, \dots, L$.

LEMMA S8: *Suppose A1', A2, and A3 hold. Suppose $L = K + 1$. Consider the hyperplanes $\{b \in \mathbb{R}^K : \pi'_{\ell}b = \psi_{\ell}\}$ for $\ell = 1, \dots, L$. There exists a normalization of these hyperplanes such that $\psi_{\ell} \geq 0$ for all $\ell = 1, \dots, L$ and such that*

$$0_K \in \text{conv}(\{\pi_{\ell} : \ell = 1, \dots, L\}) \quad \Rightarrow \quad 0_K \in \text{conv}(\{\beta_{-\ell}^{2\text{SLS}} : \ell = 1, \dots, L\}).$$

LEMMA S9: *Suppose A1', A2, and A3 hold. Suppose $L \geq K + 1$. Let $b \notin \mathcal{P}$. Then there exists a $\delta' < \delta(b)$ such that $\mathcal{B}(\delta') \neq \emptyset$.*

PROOF OF LEMMA S9: Without loss of generality, suppose $b = 0_K$. This follows since we can simply translate our coordinate system so that the origin is at b . Put differently, we map all $x \in \mathbb{R}^K$ to $x - b$. Throughout this proof, we also use a normalization from Lemma S8 where $\psi_{\ell} \geq 0$ for all ℓ . Next, there are two cases to consider.

Case 1. Suppose $\delta_{\ell}(b) = |\psi_{\ell} - \pi'_{\ell}b| = \psi_{\ell} > 0$ for all ℓ . Since $b \notin \mathcal{P}$, $b \notin \mathcal{P}_{\mathcal{L}} = \text{conv}(\{\beta_{\mathcal{L} \setminus \{\ell\}}^{2\text{SLS}} : \ell \in \mathcal{L}\})$ for any \mathcal{L} with $|\mathcal{L}| = K + 1$. Since $b = 0_K$, Lemma S8 implies that $0_K \notin \text{conv}(\{\pi_{\ell} : \ell \in \mathcal{L}\})$. This holds for any set \mathcal{L} such that $|\mathcal{L}| = K + 1$. This implies that $0_K \notin \text{conv}(\{\pi_{\ell} : \ell = 1, \dots, L\})$. To see this, assume $0_K \in \text{conv}(\{\pi_{\ell} : \ell = 1, \dots, L\})$. By Caratheodory's theorem, (e.g., Chapter 17 of Rockafellar (1970)) 0_K is then in the convex hull of a $(K + 1)$ -element subset of $\{\pi_{\ell} : \ell = 1, \dots, L\}$. That is, $0_K \in \text{conv}(\{\pi_{\ell} : \ell \in \mathcal{L}\})$ for some \mathcal{L} with $|\mathcal{L}| = K + 1$. This is a contradiction.

Since $0_K \notin \text{conv}(\{\pi_{\ell} : \ell = 1, \dots, L\})$, Lemma S5 implies that there exists a vector \bar{b} such that $\pi'_{\ell}\bar{b} > 0$ for all $\ell = 1, \dots, L$. Define $b(\varepsilon) = b + \varepsilon\bar{b}$. Since $\psi_{\ell} > 0$ and $\pi'_{\ell}\bar{b} > 0$ for all

ℓ , there exists an $\bar{\varepsilon} > 0$ such that $\psi_\ell - \bar{\varepsilon}\pi'_\ell\bar{b} > 0$ for all ℓ . This implies that

$$0 < |\psi_\ell - \pi'_\ell b(\bar{\varepsilon})| = |\psi_\ell - \pi'_\ell(\bar{\varepsilon}\bar{b})| = \delta_\ell(b) - \bar{\varepsilon}\pi'_\ell\bar{b} < \delta_\ell(b)$$

by $\delta_\ell(b) = \psi_\ell$ and by $0 < \bar{\varepsilon}\pi'_\ell\bar{b} < \psi_\ell$ for all ℓ .

Let $\delta'_\ell = |\psi_\ell - \pi'_\ell b(\bar{\varepsilon})|$. We have shown that $\delta' < \delta(b)$. Finally, by our characterization of $\mathcal{B}(\cdot)$ and the definition of δ' , we have $b(\bar{\varepsilon}) \in \mathcal{B}(\delta')$. Hence $\mathcal{B}(\delta') \neq \emptyset$.

Case 2. Suppose $\delta_\ell(b) = 0$ for some ℓ 's. There can be at most $K - 1$ such indices, since otherwise we would have $b \in \mathcal{P}$. Let \mathcal{L}_0 denote the set of these indices. Since $b = 0_K$ and $\delta_\ell(b) = 0$, $\psi_\ell = 0$ for $\ell \in \mathcal{L}_0$. In this case, consider the subspace

$$\{\tilde{b} \in \mathbb{R}^K : 0 = \pi'_\ell\tilde{b}, \ell \in \mathcal{L}_0\}.$$

This is a linear subspace of dimension at least 1 (by $|\mathcal{L}_0| > 0$) and at most $K - 1$ (as noted earlier). Within this subspace, we can look at the remaining indices $\{1, \dots, L\} \setminus \mathcal{L}_0$. We have $\delta_\ell(b) > 0$ for all of these indices. By restricting attention to this subspace we can thus immediately apply the analysis of case 1. *Q.E.D.*

For the next two lemmas, let

$$\text{FF}^{\text{guess}} = \{\delta \in \mathbb{R}_{\geq 0}^L : \delta_\ell = |\psi_\ell - \pi'_\ell b|, \ell = 1, \dots, L, b \in \mathcal{P}\}$$

and let FF denote the true falsification frontier from Definition 1.

LEMMA S10: *Suppose A1', A2, and A3 hold. Suppose $L \geq K + 1$. Then $\text{FF}^{\text{guess}} \subseteq \text{FF}$.*

PROOF OF LEMMA S10: Recall the definition $\delta_\ell(b) = |\psi_\ell - \pi'_\ell b|$. Let $\delta \in \text{FF}^{\text{guess}}$. Then, by definition, there is a $b \in \mathcal{P}$ such that $\delta_\ell(b) = \delta_\ell$ for all ℓ . Thus $\mathcal{B}(\delta) = \{b\}$ by Lemma S4.

Let $\delta' < \delta(b)$. Then there is some index ℓ such that $0 < \delta'_\ell < \delta_\ell(b)$. So $\psi_\ell - \pi'_\ell b \notin [-\delta'_\ell, \delta'_\ell]$ and hence $b \notin \mathcal{B}_\ell(\delta')$. This implies that $b \notin \mathcal{B}(\delta')$. But since $\mathcal{B}(\delta') \subseteq \mathcal{B}(\delta) = \{b\}$, we must have $\mathcal{B}(\delta') = \emptyset$. Hence, by the definition of the falsification frontier, $\text{FF}^{\text{guess}} \subseteq \text{FF}$. *Q.E.D.*

LEMMA S11: *Suppose A1', A2, and A3 hold. Suppose $L \geq K + 1$. Then $\text{FF}^{\text{guess}} \supseteq \text{FF}$.*

PROOF OF LEMMA S11: We will show the contrapositive: $\delta \notin \text{FF}^{\text{guess}}$ implies $\delta \notin \text{FF}$. Let $\delta \notin \text{FF}^{\text{guess}}$. There are two cases to consider.

Case 1. Suppose δ is such that $\mathcal{B}(\delta)$ contains an element $b \notin \mathcal{P}$. By Lemma S9, there exists $\delta' < \delta(b)$ with $\mathcal{B}(\delta') \neq \emptyset$. Moreover, $\delta(b) \leq \delta$ by the characterization of $\mathcal{B}(\delta)$ in Theorem 1. Thus $\delta \notin \text{FF}$ by the definition of the falsification frontier.

Case 2. Suppose δ is such that $\mathcal{B}(\delta) \subseteq \mathcal{P}$. If $\mathcal{B}(\delta) = \emptyset$, then $\delta \notin \text{FF}$ by definition. Therefore we let $\mathcal{B}(\delta) \neq \emptyset$. Let b' be any element of $\mathcal{B}(\delta)$. Let $\delta' = \delta(b')$. By $b' \in \mathcal{P}$, $\delta' \in \text{FF}^{\text{guess}}$ and by $\delta \notin \text{FF}^{\text{guess}}$, $\delta' \neq \delta$. Also, by $b' \in \mathcal{B}(\delta)$ we have $\delta'_\ell = |\psi_\ell - \pi'_\ell b'| \leq \delta_\ell$ for all ℓ . Together these imply $\delta' < \delta$. Moreover, we have $\mathcal{B}(\delta') = \mathcal{B}(\delta(b')) = \{b'\} \neq \emptyset$ by Lemma S4. Thus $\delta \notin \text{FF}$, by definition of the falsification frontier.

All values of δ must fall in one of these two cases. Therefore $\text{FF}^{\text{guess}} \supseteq \text{FF}$. *Q.E.D.*

PROOF OF PROPOSITION S1: This follows directly from Lemmas S10 and S11. *Q.E.D.*

PROOF OF THEOREM S1: We have

$$\bigcup_{\delta \in \text{FF}} \mathcal{B}(\delta) = \bigcup_{b \in \mathcal{P}} \mathcal{B}(\delta(b)) = \bigcup_{b \in \mathcal{P}} \{b\} = \mathcal{P}$$

by Proposition S1 and Lemma S4.

Q.E.D.

To prove Corollary S1, we use the following definition: Let P be a $K \times K$ matrix. Define the linear operator $p : \mathbb{R}^K \rightarrow \mathbb{R}^K$ by $p(a) = Pa$. For $A \subseteq \mathbb{R}^K$, define the set $\text{proj}(A) = \{p(a) \in \mathbb{R}^K : a \in A\}$.

We use the following lemma in the proof of the corollary.

LEMMA S12: $\text{proj}(\text{conv}(A)) = \text{conv}(\text{proj}(A))$.

PROOF OF COROLLARY S1: The first part of this result states that $\mathcal{P} \subseteq \text{FAS}^*$. To see this, let $b \in \mathcal{P}$. Then $b \in \mathcal{P}_{\mathcal{L}}$ for some set of indices \mathcal{L} with $|\mathcal{L}| = K + 1$. But if b is a convex combination of $\{\beta_{\mathcal{L} \setminus \{\ell\}}^{2\text{SLS}} : \ell \in \mathcal{L}\}$, then it is also a convex combination of the larger set of elements $\{\beta_{\mathcal{S}}^{2\text{SLS}} : \mathcal{S} \subseteq \{1, \dots, L\}, |\mathcal{S}| = K\}$. Hence $b \in \text{FAS}^*$.

To prove the second part, consider the $K \times K$ matrix

$$P = (\alpha \quad 0_K \quad \cdots \quad 0_K)',$$

where $\alpha \in \mathbb{R}^K$. For any set $A \subseteq \mathbb{R}^K$, let $[A]_1 = \{a_1 \in \mathbb{R} : a = (a_1, \dots, a_K) \in A\}$. Since P maps the 2, ..., K components of any vector $a \in \mathbb{R}^K$ to zero, it suffices to show that $[\text{proj}(\mathcal{P})]_1 = [\text{proj}(\text{FAS}^*)]_1$ where

$$[\text{proj}(\text{FAS}^*)]_1 = \left[\min_{\mathcal{L} \subseteq \{1, \dots, L\}: |\mathcal{L}|=K} \alpha' \beta_{\mathcal{L}}^{2\text{SLS}}, \max_{\mathcal{L} \subseteq \{1, \dots, L\}: |\mathcal{L}|=K} \alpha' \beta_{\mathcal{L}}^{2\text{SLS}} \right]. \quad (\text{S3})$$

We have

$$\text{proj}(\text{FAS}^*) = \text{proj}(\text{conv}(\mathcal{P})) = \text{conv}(\{P\beta_{\mathcal{L}}^{2\text{SLS}} : \mathcal{L} \subseteq \{1, \dots, L\}, |\mathcal{L}| = K\})$$

by Lemma S12. Using the specific form of P now gives equation (S3).

Similarly,

$$\begin{aligned} \text{proj}(\mathcal{P}) &= \bigcup_{\mathcal{L} \subseteq \{1, \dots, L\}: |\mathcal{L}|=K+1} \text{proj}(\text{conv}(\{\beta_{\mathcal{L} \setminus \{\ell\}}^{2\text{SLS}} : \ell \in \mathcal{L}\})) \\ &= \bigcup_{\mathcal{L} \subseteq \{1, \dots, L\}: |\mathcal{L}|=K+1} \text{conv}(\text{proj}(\{\beta_{\mathcal{L} \setminus \{\ell\}}^{2\text{SLS}} : \ell \in \mathcal{L}\})). \end{aligned}$$

The first line follows by $\text{proj}(A \cup B) = \text{proj}(A) \cup \text{proj}(B)$ for any sets $A, B \subseteq \mathbb{R}^K$. The second line follows from Lemma S12. Hence

$$[\text{proj}(\mathcal{P})]_1 = \bigcup_{\mathcal{L} \subseteq \{1, \dots, L\}: |\mathcal{L}|=K+1} \left[\min_{\ell \in \mathcal{L}} \alpha' \beta_{\mathcal{L} \setminus \{\ell\}}^{2\text{SLS}}, \max_{\ell \in \mathcal{L}} \alpha' \beta_{\mathcal{L} \setminus \{\ell\}}^{2\text{SLS}} \right].$$

Thus we see that the first component of $\text{proj}(\mathcal{P})$ is a union of closed intervals. If \mathcal{L} and \mathcal{L}' differ by at most one element, then the intervals

$$\left[\min_{\ell \in \mathcal{L}} \alpha' \beta_{\mathcal{L} \setminus \{\ell\}}^{2\text{SLS}}, \max_{\ell \in \mathcal{L}} \alpha' \beta_{\mathcal{L} \setminus \{\ell\}}^{2\text{SLS}} \right] \quad \text{and} \quad \left[\min_{\ell \in \mathcal{L}'} \alpha' \beta_{\mathcal{L}' \setminus \{\ell\}}^{2\text{SLS}}, \max_{\ell \in \mathcal{L}'} \alpha' \beta_{\mathcal{L}' \setminus \{\ell\}}^{2\text{SLS}} \right]$$

have nonempty intersection. Their union is therefore a closed interval. Because we take the union over all $\mathcal{L} \subseteq \{1, \dots, L\}$ such that $|\mathcal{L}| = K + 1$, we can find a sequence $(\mathcal{L}_1, \dots, \mathcal{L}_N)$ such that

$$\left[\min_{\ell \in \mathcal{L}_n} \alpha' \beta_{\mathcal{L}_n \setminus \{\ell\}}^{2SLS}, \max_{\ell \in \mathcal{L}_n} \alpha' \beta_{\mathcal{L}_n \setminus \{\ell\}}^{2SLS} \right] \quad \text{and} \quad \left[\min_{\ell \in \mathcal{L}_{n+1}} \alpha' \beta_{\mathcal{L}_{n+1} \setminus \{\ell\}}^{2SLS}, \max_{\ell \in \mathcal{L}_{n+1}} \alpha' \beta_{\mathcal{L}_{n+1} \setminus \{\ell\}}^{2SLS} \right]$$

overlap for $n = 1, \dots, N - 1$ and such that $\bigcup_{n=1}^N \mathcal{L}_n = \{1, \dots, L\}$. Thus

$$\bigcup_{\mathcal{L} \subseteq \{1, \dots, L\}; |\mathcal{L}|=K+1} \left[\min_{\ell \in \mathcal{L}} \alpha' \beta_{\mathcal{L} \setminus \{\ell\}}^{2SLS}, \max_{\ell \in \mathcal{L}} \alpha' \beta_{\mathcal{L} \setminus \{\ell\}}^{2SLS} \right] = \left[\min_{\mathcal{L} \subseteq \{1, \dots, L\}; |\mathcal{L}|=K} \alpha' \beta_{\mathcal{L}}^{2SLS}, \max_{\mathcal{L} \subseteq \{1, \dots, L\}; |\mathcal{L}|=K} \alpha' \beta_{\mathcal{L}}^{2SLS} \right].$$

Putting everything together yields $[\text{proj}(\mathcal{P})]_1 = [\text{proj}(\text{FAS}^*)]_1$ as desired.

Q.E.D.

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