# Erratum for "Identification of Treatment Effects under Conditional Partial Dependence"

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Proposition 5 on page 329 provides bounds for $\mathbb{P}(Y_x = 1 \mid W = w)$. While the provided bounds (see the expressions for $(\underline{P}^c_x(1 \mid w), \overline{P}^c_x(1 \mid w))$ on page 329) are correct bounds for $\mathbb{P}(Y_x = 1 \mid W = w)$, they are in fact conservative and hence not sharp. In this erratum we provide new expressions for $(\underline{P}^c_x(1 \mid w), \overline{P}^c_x(1 \mid w))$ that are narrower than the original bounds, and show that these new bounds are sharp in the sense originally stated in our Proposition 5.

## New Bounds Expressions

Let

$$\overline{P}^c_x(y \mid w) = \min\left\{\frac{p_{y|x,w}p_{x|w}}{p_{x|w} - c}\mathbb{1}(p_{x|w} > c) + \mathbb{1}(p_{x|w} \leq c), \frac{p_{y|x,w}p_{x|w} + c}{p_{x|w} + c}, p_{y|x,w}p_{x|w} + 1 - p_{x|w}\right\}$$

$$\underline{P}^c_x(y \mid w) = \max\left\{\frac{p_{y|x,w}p_{x|w}}{p_{x|w} + c}, \frac{p_{y|x,w}p_{x|w} - c}{p_{x|w} - c}\mathbb{1}(p_{x|w} > c), p_{y|x,w}p_{x|w}\right\}.$$

Set $y = 1$ to replace the original bounds on page 329. Note that these news bounds are weakly narrower than the original ones.

## Sharpness Proof

We now show that these new narrower bounds are sharp. To do so, we amend the proof of Proposition 5. The proof is split into two parts: (1) showing that $[\underline{P}^c_x(1 \mid w), \overline{P}^c_x(1 \mid w)]$ are bounds for $\mathbb{P}(Y_x = 1 \mid W = w)$, and (2) showing sharpness of the interior of this interval.

### Modification of Part (1)

The original proof shows that the original expressions are bounds for $\mathbb{P}(Y_x = 1 \mid W = w)$. To show the new expressions are also bounds for $\mathbb{P}(Y_x = 1 \mid W = w)$, it is sufficient to show that

$$\mathbb{P}(Y_x = 1 \mid W = w) \in \left[\frac{p_{1|x,w}p_{x|w} - c}{p_{x|w} - c}\mathbb{1}(p_{x|w} > c), \frac{p_{1|x,w}p_{x|w} + c}{p_{x|w} + c}\right].$$

This is obtained as follows:

$$\begin{aligned}\mathbb{P}(Y_x = 1 \mid W = w) &= 1 - \mathbb{P}(Y_x = 0 \mid W = w) \\ &= 1 - \frac{p_{0|x,w}p_{x|w}}{\mathbb{P}(X = x \mid Y_x = 0, W = w)}\end{aligned}$$

$$\in \left[1 - \frac{p_{0|x,w}p_{x|w}}{p_{x|w} - c}\mathbb{1}(p_{x|w} > c) - \mathbb{1}(p_{x|w} \le c), 1 - \frac{p_{0|x,w}p_{x|w}}{p_{x|w} + c}\right]$$
$$= \left[\frac{p_{1|x,w}p_{x|w} - c}{p_{x|w} - c}\mathbb{1}(p_{x|w} > c), \frac{p_{1|x,w}p_{x|w} + c}{p_{x|w} + c}\right].$$

The third line follows by conditional $c$-dependence. □

**Modification of Part (2)**

To amend the sharpness proof, we show that

$$\mathbb{P}(X = 1 \mid Y_x = y, W = w) \in [p_{1|w} - c, p_{1|w} + c]$$

when

$$\mathbb{P}(Y_x = 1 \mid x = 1 - x, W = 1) = \frac{p^* - p_{1|x,w}p_{x|w}}{1 - p_{x|w}}$$

for any $p^* \in (\underline{P}^c_x(1 \mid w), \overline{P}^c_x(1 \mid w))$ and for $x, y \in \{0, 1\}$. The following replaces "*Proof of 3.*" on pages 349–350.

From Bayes' rule we have

$$\mathbb{P}(X = x \mid Y_x = y, W = w) = \frac{p_{y|x,w}p_{x|w}}{\mathbb{P}(Y_x = y \mid W = w)} \in \left[\frac{p_{y|x,w}p_{x|w}}{\overline{P}^c_x(y \mid w)}, \frac{p_{y|x,w}p_{x|w}}{\underline{P}^c_x(y \mid w)}\right].$$

The inclusion follows by validity of the bounds, from part (1). Substituting in the new expressions for $(\underline{P}^c_x(y \mid w), \overline{P}^c_x(y \mid w))$ we find

$$\begin{aligned}
\frac{p_{y|x,w}p_{x|w}}{\underline{P}^c_x(y \mid w)} &= \frac{p_{y|x,w}p_{x|w}}{\max\left\{\frac{p_{y|x,w}p_{x|w}}{p_{x|w}+c}, \frac{p_{y|x,w}p_{x|w}-c}{p_{x|w}-c}\mathbb{1}(p_{x|w} > c), p_{y|x,w}p_{x|w}\right\}} \\
&\le p_{y|x,w}p_{x|w} \Bigg/ \frac{p_{y|x,w}p_{x|w}}{p_{x|w} + c} \\
&= p_{x|w} + c
\end{aligned}$$

and

$$\begin{aligned}
\frac{p_{y|x,w}p_{x|w}}{\overline{P}^c_x(y \mid w)} &= \frac{p_{y|x,w}p_{x|w}}{\min\left\{\frac{p_{y|x,w}p_{x|w}}{p_{x|w}-c}\mathbb{1}(p_{x|w} > c) + \mathbb{1}(p_{x|w} \le c), \frac{p_{y|x,w}p_{x|w}+c}{p_{x|w}+c}, p_{y|x,w}p_{x|w} + 1 - p_{x|w}\right\}} \\
&\ge \frac{p_{y|x,w}p_{x|w}}{\frac{p_{y|x,w}p_{x|w}}{p_{x|w}-c}\mathbb{1}(p_{x|w} > c) + \mathbb{1}(p_{x|w} \le c)} \\
&= (p_{x|w} - c)\mathbb{1}(p_{x|w} > c) + p_{y|x,w}p_{x|w}\mathbb{1}(p_{x|w} \le c) \\
&\ge p_{x|w} - c.
\end{aligned}$$

Therefore $\mathbb{P}(X = x \mid Y_x = y, W = w) \in [p_{x|w} - c, p_{x|w} + c]$ for $x, y \in \{0, 1\}$. This implies that

$$\mathbb{P}(X = 1 \mid Y_1 = y, W = w) \in [p_{1|w} - c, p_{1|w} + c]$$

and that

$$\begin{aligned}
\mathbb{P}(X = 1 \mid Y_0 = y, W = w) &= 1 - \mathbb{P}(X = 0 \mid Y_0 = y, W = w) \\
&\in [1 - (p_{0|w} + c), 1 - (p_{0|w} - c)] \\
&= [p_{1|w} - c, p_{1|w} + c]
\end{aligned}$$

for $y \in \{0, 1\}$. □